ON THE STEADY-STATE OSCILLATIONS OF A PLATE SIMPLY SUPPORTED ALONG THE EDGE

(O STATSIONARNYKH KOLEBANIIAKH PLASTINKI, OPERTOI PO KRAIU)

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We consider the steady-state transverse oscillations of a homogeneous isotropic plate of constant thickness which occupies an arbitrary simply connected region. The edges of the plate are assumed to be simply supported. The general case of static bending of a plate occupying an arbitrary (simply or multiply connected) region was studied before by Sherman [1,2]. The representations of the unknown regular functions proposed by him were successfully applied in the final stage of the present investigation of steady-state oscillations. This way, the problem was reduced to the solution of a Fredholm's integral equation of the second kind.

1. Let the middle surface of an oscillating plate occupy an arbitrary simply connected region S in the complex plane $z = x + iy$; the origin is assumed to be located within the region S. Moreover, we assume that the contour L, which bounds the region, has a differentiable curvature.

In order to find the amplitude of the oscillations $u(x, y)$, it is necessary to solve the following differential equation in the region S:

$$\triangle\triangle u - \lambda^4 u = \frac{q(x, y)}{D} \qquad \left(\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \ \lambda = \left(\frac{\rho h \nu^2}{D}\right)^{1/4}\right) \tag{1.1}$$

Here, $q(x, y)$ and ν are the amplitude and frequency of oscillations of the transverse load, ρ is the density, h and D are, respectively, the thickness and the flexural rigidity of the plate.

Besides equation (1.1), the amplitude $u(x, y)$ must also satisfy the boundary conditions on the contour L

$$u = 0, \qquad \frac{\sigma\triangle u}{1-\sigma} + \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial x\,\partial y}\sin 2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta = 0 \tag{1.2}$$

Here, σ is Poisson's ratio and θ is the angle between the outward normal to the contour and the abscissa.

Let us differentiate the first boundary condition (1.2) twice with respect to the contour arc s and add it to the second condition, term by term. Let us join the new boundary condition thus obtained and the first boundary condition (1.2); we take this set as the new modified boundary conditions (obviously, equivalent to the original ones).

In the complex form they are written down as follows*:

$$u = 0, \qquad G(u) \equiv \varkappa \frac{\partial^2 u}{\partial t\,\partial \bar{t}} + t'' \frac{\partial u}{\partial t} + \overline{t''}\frac{\partial u}{\partial \bar{t}} = 0 \quad \left(\varkappa = \frac{4}{1-\sigma}\right) \tag{1.3}$$

where t is the complex coordinate of a point on the contour, the primes designate differentiation with respect to the arc s.

Furthermore, we introduce a new function $w(x, y)$ by the formula

$$u(x, y) = w(x, y) + w_0(x, y) - [w(a) + w_0(a)]\,J_0(\lambda r_a) \tag{1.4}$$
$$\left(r_a = \sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$$

Here, $w_0(x, y)$ is the particular solution of the nonhomogeneous equation (1.1) and $J_0(\lambda r_a)$ is the Bessel function of order zero; moreover, $a = x_0 + iy_0$ is the affix of the fixed point of the contour from which the arc length is measured; $w(a)$ and $w_0(a)$ are the values of functions $w(x, y)$ and $w_0(x, y)$ at point a. Instead of the first boundary equality (1.3) let us take the following condition:

$$\partial u / \partial s + \operatorname{Re}\varphi'(0) = 0 \tag{1.5}$$

where $\varphi'(0)$ is the value of the derivative at point $z = 0$ of a certain function $\varphi(z)$ (defined below), regular in the region S. The following condition ensues from relation (1.5) [1,2]:

$$\operatorname{Re}\varphi'(0) = 0 \tag{1.6}$$

Indeed, the first term of relation (1.5) is the total differential with respect to the arc s of a certain continuous and single-valued function; therefore, integrating relation (1.5) with respect to the arc s along the closed contour L, and bearing in mind that the function

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* A similar modification of the boundary conditions was used in the solution of the static problem of bending of a simply supported plate [1,2]. The purpose of such a modification is explained in [2].

$u(x, y)$ is single-valued, we arrive at condition (1.6). Let us note that the fulfilment of the last condition will guarantee the solvability of the integral equation which will be obtained below.

We substitute the value of the amplitude $u(x, y)$ from (1.4) into equation (1.1), the boundary relation (1.5) and the second condition (1.3). As a result, we obtain a homogeneous equation for the auxiliary function $w(x, y)$

$$\Delta\Delta w - \lambda^4 w = 0 \tag{1.7}$$

with boundary conditions

$$\partial w / \partial s + \lambda w(a) J_1(\lambda r_a) \cos\Omega + \operatorname{Re}\varphi'(0) = f_1(t) \tag{1.8}$$

$$G(w) - w(a)\, G[J_0(\lambda r_a)] = f_2(t) \tag{1.9}$$

Here

$$f_1(t) = -\partial w_0 / \partial s - \lambda w_0(a) J_1(\lambda r_a) \cos\Omega, \quad f_2(t) = -G(w_0) + w_0(a)\, G[J_0(\lambda r_a)]$$

(Ω is the angle between the vector $t - a$ and the tangent vector to the contour L at point t oriented in the positive direction of the contour line). Obviously, the function $u(x, y)$ expressed in terms of $w(x, y)$ by means of relation (1.4) is the solution of the original problem. Indeed, by virtue of (1.7) and (1.9), $u(x, y)$ satisfies the nonhomogeneous equation (1.1) and the second boundary condition (1.3). Moreover, it follows from relation (1.5) that $u(x, y)$ has a constant value on the contour L. In so far as the right-hand side of relation (1.4) becomes equal to zero at point $z = a$, we conclude that the function $u(x, y)$ will also satisfy the first boundary condition (1.3).

Note. In the solution of the static problem of bending of a plate, the first condition (1.3) may be satisfied up to an arbitrary constant. Therefore, it can be replaced by the condition that the derivative of the deflection with respect to the contour arc s be equal to zero. Such modification leads to a change of solution by a constant in the entire region, which corresponds to the rigid displacement of the plate in the direction of its normal. However, in the problem of forced oscillations it is necessary to ensure the exact fulfilment of both conditions (1.3); in that case, the change of the first boundary condition (1.3) by a constant results in a change of the solution by a function different from constant within the region S. In order to avoid that, one has to introduce an auxiliary function $w(x, y)$ by means of a specially chosen relation (1.4).

2. Let us proceed to the determination of the function $w(x, y)$. First we combine the two real-valued boundary conditions (1.8) and (1.9) into one complex equality as follows:

$$\frac{\partial w}{\partial s} + iG(w) + w(a) f_3(t) + \operatorname{Re} \varphi'(0) = \frac{1}{2} f(t) \tag{2.1}$$

Here

$$f_3(t) = \lambda J_1(\lambda r_a) \cos \Omega - iG[J_0(\lambda r_a)], \qquad f(t) = 2[f_1(t) + if_2(t)] \tag{2.2}$$

Let us use the general representation of the solutions (1.7) given by Vekua [3]

$$w(x, y) = \operatorname{Im} \Big\{ \chi(z) - z\bar{z} \int_0^1 \chi(z\tau) P_0[z\bar{z}(1-\tau)]\, d\tau +$$

$$+ \bar{z}\varphi(z) - z\bar{z}^2 \int_0^1 \varphi(z\tau) P_1[z\bar{z}(1-\tau)]\, d\tau \Big\} \tag{2.3}$$

$$P_0(x) = \frac{\lambda}{4} \frac{J_1(\lambda\sqrt{x}) - I_1(\lambda\sqrt{x})}{\sqrt{x}}, \qquad P_1(x) = \frac{J_2(\lambda\sqrt{x}) - I_2(\lambda\sqrt{x})}{2x}$$

Here, $\varphi(z)$ and $\chi(z)$ are arbitrary functions, regular in the region S, $J_i(\lambda\sqrt{x})$ and $I_i(\lambda\sqrt{x})$ are, respectively, ordinary and modified Bessel functions ($i = 1, 2$).

Furthermore, we substitute the derivatives $\partial w/\partial z$, $\partial w/\partial \bar{z}$, $\partial^2 w/\partial z \partial \bar{z}$ from (2.3) into the boundary condition (2.1); we replace the functions $P_0[z\bar{z}(1-\tau]$ and $P_1[z\bar{z}(1-\tau)]$ by their expansions into series. Thus we arrive at the following boundary value problem for the two functions $\varphi(z)$ and $\chi(z)$, regular in the region S:

$$\varkappa[\varphi'(t) - \overline{\varphi'(t)}] + \alpha(t)\gamma(t) - \beta(t)\overline{\gamma(t)} + 2\operatorname{Re}\varphi'(0) + w(a) f_3(t) + \tag{2.4}$$

$$+ \int_0^1 [\chi(t\tau) K_1(t, \tau) + \overline{\chi(t\tau)} K_2(t, \tau) + \varphi(t\tau) K_3(t, \tau) + \overline{\varphi(t\tau)} K_4(t, \tau)]\, d\tau = f(t)$$

In this boundary condition, the following designations were used:

$$\alpha(t) = \bar{t}'' - i\bar{t}', \quad \beta(t) = t'' - it', \quad \gamma(t) = \varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}, \quad \chi'(t) = \psi(t)$$

$$K_i(t, \tau) = \lambda^2 \sum_{k=0}^{\infty} \frac{(1-\tau)^k}{(k!)^2} A_{ik}(t) \qquad (i = 1, 2, 3, 4)$$

$$A_{1k}(t) = \left[\varkappa + \frac{\beta(t)\bar{t}}{k+2}\right] b_k(t) + a_k(t)\alpha(t), \qquad a_k(t) = \frac{t}{8}\left(\frac{\lambda^2 t\bar{t}}{4}\right)^k [1 - (-1)^k]$$

$$A_{2k}(t) = -\left[\varkappa + \frac{\alpha(t)t}{k+2}\right] b_k(t) - \overline{a_k(t)}\beta(t), \qquad b_k(t) = \frac{1}{8}\left(\frac{\lambda^2 t\bar{t}}{4}\right)^{k+1} [1 + (-1)^k]$$

$$A_{3k}(t) = \bar{t}\left\{\left[\varkappa + \frac{\bar{t}\beta(t)}{k+3}\right]\frac{b_k(t)}{k+2} + \frac{a_k(t)\alpha(t)}{k+1}\right\}$$

$$A_{4k}(t) = -t\left\{\left[\varkappa + \frac{t\alpha(t)}{k+3}\right]\frac{b_k(t)}{k+2} + \frac{\overline{a_k(t)}\,\beta(t)}{k+1}\right\}$$

It is easy to see that the series $K_i(t, \tau)$, under the integral sign, are absolutely and uniformly convergent, so that $K_i(t, \tau)$ are continuous functions of t and τ. It is a favorable circumstance in this case, that all the extra-integral terms of the boundary condition (2.4) are generated exclusively by the extra-integral terms of the representation (2.3). At the same time it is easy to observe that the extra-integral terms of expression (2.3) coincide with Goursat's representation for the biharmonic function. As a result, the first three terms of the boundary value problem (2.4) formally coincide with the corresponding terms of relation (1.7) in [1]. Thus, in the solution of the boundary value problem (2.4) one can use representations for the functions $\varphi(z)$ and $\psi(z)$ contained in [1]

$$\varphi(z) = \int_L [\omega(\xi)\theta_1(\xi) + \overline{\omega(\xi)}\theta_2(\xi)]\, F(\xi, z)\, d\xi \tag{2.5}$$

$$\chi(z) = \int_L [\omega(\xi) R_1(\xi, z) + \overline{\omega(\xi)} R_2(\xi, z)]\, d\xi \tag{2.6}$$

$$\psi(z) = \int_L [\omega(\xi) H_1(\xi, z) + \overline{\omega(\xi)} H_2(\xi, z)]\, d\xi \tag{2.7}$$

Here, $\omega(\xi)$ is the density which has to be determined. The meaning of the other functions contained in (2.5), (2.6) and (2.7) is as follows:

$$\theta_1(\xi) = i + \overline{\xi}'\xi'', \qquad \theta_2(\xi) = -i + \overline{\xi}'\xi'', \qquad F(\xi, z) = \frac{1}{4\pi\varkappa}\left[-1 + \ln\left(1 - \frac{z}{\xi}\right)\right]$$

$$R_1(\xi, z) = -\overline{\xi'^2}\,\overline{\theta_2(\xi)}\,(\xi - z)\, F(\xi, z) + P(\xi)\, T(\xi, z)$$

$$R_2(\xi, z) = -\overline{\xi'^2}\,\overline{\theta_1(\xi)}\,(\xi - z)\, F(\xi, z) + Q(\xi)\, T(\xi, z)$$

$$H_1(\xi, z) = \frac{\overline{\xi'^2}\,\overline{\theta_2(\xi)}}{4\pi\varkappa}\ln\left(1 - \frac{z}{\xi}\right) + P(\xi)\left(\frac{1}{\xi - z} - \frac{1}{\xi}\right), \qquad P(\xi) = \frac{1}{4\pi\varkappa}[\varkappa\overline{\xi}' - \overline{\xi}\,\overline{\theta_1(\xi)}]$$

$$H_2(\xi, z) = \frac{\overline{\xi'^2}\,\overline{\theta_1(\xi)}}{4\pi\varkappa}\ln\left(1 - \frac{z}{\xi}\right) + Q(\xi)\left(\frac{1}{\xi - z} - \frac{1}{\xi}\right), \qquad Q(\xi) = \frac{1}{4\pi\varkappa}[\varkappa\overline{\xi}' - \overline{\xi}\,\overline{\theta_2(\xi)}]$$

$$T(\xi, z) = -\ln\left(1 - \frac{z}{\xi}\right) - \frac{z}{\xi}$$

Under $\ln(1 - z\xi^{-1})$ we understand a branch which becomes equal to zero for $z = 0$.

In the representations (2.5), (2.6) and (2.7) and in the expression for $\varphi'(z)$ we let $z \to t$, where t is a point on the contour L; we substitute the obtained limiting values into equality (2.4), also changing the order of integration. The form of the representations (2.5), (2.6) and (2.7) turns out to be such that, upon their substitution into the boundary relation (2.4), the ensuing combinations of integrals in the sense of principal value according to Cauchy, lead, as a final result, to regular integrals; therefore, for the determination of the density $\omega(t)$ we obtain a Fredholm's integral equation of the second kind with continuous kernels

$$\omega(t)+\int_L\{\omega(\xi)\,[M(\xi,t)+M_1(\xi,t)]+\overline{\omega(\xi)}\,[N(\xi,t)+N_1(\xi,t)]\}\,d\xi+ \\ +O\,[\omega(\xi),t]+w\,[\omega(\xi),a]\,f_3(t)=f(t) \tag{2.8}$$

The functions $M(\xi,t)$ and $N(\xi,t)$ appearing here, completely coincide with the kernels of the integral equation of the statical problem for a simply connected region, and the operator $O[\omega(\xi),t]$ coincides with the corresponding operator of the statical problem [1]

$$M(\xi,t)=v_1(\xi,t)+l_1(\xi,t)\frac{d}{d\xi}\ln\frac{\xi-t}{\bar\xi-\bar t}+\frac{p(\xi,t)}{\xi-t}+q_1(\xi,t) \tag{2.9}$$

$$N(\xi,t)=v_2(\xi,t)+l_2(\xi,t)\frac{d}{d\xi}\ln\frac{\xi-t}{\bar\xi-\bar t}+\frac{p(\xi,t)}{\xi-t}+q_2(\xi,t)$$

$$v_1(\xi,t)=\overline{t'}\,\overline{\theta_1(t)}\,a(\xi,t)-\overline{\xi'^2}t'\theta_2(t)\,\overline{b(\xi,t)}$$

$$v_2(\xi,t)=\overline{t'}\,\overline{\theta_1(t)}\,b(\xi,t)-\overline{\xi'^2}t'\theta_2(t)\,\overline{a(\xi,t)}$$

$$\frac{4\pi\varkappa}{\theta_1(\xi)}a(\xi,z)=-1+\ln\frac{1-(z/\xi)}{\overline{1-(z/\xi)}},\qquad b(\xi,z)=a(\xi,z)\frac{\theta_2(\xi)}{\theta_1(\xi)}$$

$$l_1(\xi,t)=\frac{1}{4\pi}\{-\overline{\theta_2(\xi)}+\overline{t'}\,\overline{\theta_1(t)}\,[\xi'-d(\xi,t)]-t'\theta_2(t)\,\overline{c(\xi,t)}\}$$

$$l_2(\xi,t)=\frac{1}{4\pi}\{-\overline{\theta_1(\xi)}+\overline{t'}\,\overline{\theta_1(t)}\,[\xi'-c(\xi,t)]-t'\theta_2(t)\,\overline{d(\xi,t)}\}$$

$$p(\xi,t)=\frac{1}{4\pi}\{\xi'\,[\overline{\xi'}\,\overline{\theta_1(\xi)}-\overline{t'}\,\overline{\theta_1(t)}]-\overline{\xi'}\,[\xi'\theta_2(\xi)-t'\theta_2(t)]\} \tag{2.10}$$

$$c(\xi,t)=\frac{1}{\varkappa}(\xi-t)\theta_1(\xi),\qquad d(\xi,t)=\frac{1}{\varkappa}(\xi-t)\theta_2(\xi)$$

$$q_1(\xi,t)=\frac{1}{4\pi\varkappa}[\overline{t'}\,\overline{\theta_1(t)}\,\theta_2(\xi)-t'\overline{\xi'^2}\theta_2(t)\,\overline{\theta_1(\xi)}]$$

$$q_2(\xi,t)=\frac{1}{4\pi\varkappa}[\overline{t'}\,\overline{\theta_1(t)}\,\theta_1(\xi)-t'\overline{\xi'^2}\theta_2(t)\,\overline{\theta_2(\xi)}]$$

$$O\,[\omega(\xi),t]=E\overline{t'}\,\overline{\theta_1(t)}-\bar E t'\theta_2(t)+2\operatorname{Re}\varphi'\,[0,\omega(\xi)]$$

$$E=\frac{1}{4\pi\varkappa}\int_L\{\omega(\xi)\,[\varkappa\xi'-\xi\theta_2(\xi)]+\overline{\omega(\xi)}\,[\varkappa\xi'-\xi\theta_1(\xi)]\}\,\frac{\overline{\xi'^2}d\xi}{\xi}$$

It is clear that if the curvature of the contour L is differentiable, the function $p(\xi,t)$ (2.10) for $\xi=t$ has a zero of the first order, and the third terms in (2.9) are bounded functions of ξ and t.

The additional terms $M_1(\xi, t)$ and $N_1(\xi, t)$ are given by the following rapidly convergent series:

$$M_1(\xi, t) = \lambda^2 \sum_{k=0}^{\infty} \frac{1}{(k!)^2} [A_{1k}(t) B_k^{(1)}(\xi, t) + A_{2k}(t) \overline{\xi'^2} \overline{B_k^{(2)}(\xi, t)} +$$
$$+ A_{3k}(t) \theta_1(\xi) B_k^{(3)}(\xi, t) + A_{4k}(t) \overline{\xi'^2} \overline{\theta_2(\xi)} \overline{B_k^{(3)}(\xi, t)}] \tag{2.11}$$

$$N_1(\xi, t) = \lambda^2 \sum_{k=0}^{\infty} \frac{1}{(k!)^2} [A_{1k}(t) B_k^{(2)}(\xi, t) + A_{2k}(t) \overline{\xi'^2} \overline{B_k^{(1)}(\xi, t)} +$$
$$+ A_{3k}(t) \theta_2(\xi) B_k^{(3)}(\xi, t) + A_{4k}(t) \overline{\xi'^2} \overline{\theta_1(\xi)} \overline{B_k^{(3)}(\xi, t)}] \tag{2.12}$$

The values of $B_k^{(i)}(\xi, t)$ $(i = 1, 2, 3)$ are given by the integrals

$$B_k^{(1)}(\xi, t) = \int_0^1 (1-\tau)^k R_1(\xi, t\tau)\, d\tau, \qquad B_k^{(2)}(\xi, t) = \int_0^1 (1-\tau)^k R_2(\xi, t\tau)\, d\tau$$
$$B_k^{(3)}(\xi, t) = \int_0^1 (1-\tau)^k F(\xi, t\tau)\, d\tau \tag{2.13}$$

Carrying out the integration with respect to τ in (2.13) we will have

$$B_k^{(1)}(\xi, t) = \overline{\theta_2(\xi)}\, \Phi_k(\xi, t) - P(\xi)\, Z_k(\xi, t)$$

$$B_k^{(2)}(\xi, t) = \overline{\theta_1(\xi)}\, \Phi_k(\xi, t) - Q(\xi)\, Z_k(\xi, t), \qquad B_k^{(3)}(\xi, t) = \frac{1}{4\pi\varkappa}\left[B_k(\xi, t) - \frac{1}{k+1}\right]$$

and the quantities which appear here are

$$\Phi_k(\xi, t) = -\overline{\xi'^2}\left\{B_k^{(3)}(\xi, t)(\xi - t) + tB_{k+1}^{(3)}(\xi, t)]\right\}$$

$$Z_k(\xi, t) = B_k(\xi, t) + \frac{t}{\xi}\,\frac{1}{(k+1)(k+2)}$$

$$B_k(\xi, t) = \left(1 - \frac{\xi}{t}\right)^{k+1}\left\{\frac{1}{k+1}\ln\left(1 - \frac{t}{\xi}\right) - \sum_{j=0}^{k} \frac{C_k^j}{(j+1)^2}\left[(-1)^j + \left(\frac{t}{\xi} - 1\right)^{-j-1}\right]\right\}$$

The functional $w[\omega(\xi), a]$ (equal to the value of function $w(x, y)$ at point $z = a$) contained in (2.8) is defined by formula (2.3) for $z = a$.

Let us now discuss the solvability of the integral equation (2.8). Following the method shown in [4], we will utilize the proof of solvability of the integral equation for the corresponding statical problem given in [1]. It follows from the relations for $M_1(\xi, t)$ and $N_1(\xi, t)$ given above, that the kernels of equation (2.8) are integral functions of the parameter λ. For $\lambda = 0$ the functions $M_1(\xi, t)$, $N_1(\xi, t)$ and $f_3(t)$

vanish and the integral equation (2.8) becomes coincident with the equation of static bending of a plate mentioned above. Hence, by virtue of the theorem of Tamarkin [5], there follows the solvability of the integral equation (2.8) for almost all values of the parameter λ.

In the case where λ does not coincide with the pole of the resolvent of equation (2.8), we first find the density $\omega(t)$ from (2.8); then from relations (2.5), (2.6), (2.7) and (2.3) we successively determine the functions $\varphi(z)$, $\chi(z)$, $\psi(z)$ and $w(x, y)$.

3. The particular solution $w_0(x, y)$ of the nonhomogeneous equation (1.1), containing an arbitrary differentiable function $q(x, y)$ on the right-hand side, can be taken in the following form:

$$w_0(x, y) = -\frac{1}{4\pi\lambda^2 D}\iint_S \left[\frac{\pi}{2} Y(\lambda r) + K(\lambda r)\right] q(\xi, \eta)\, d\xi\, d\eta \tag{3.1}$$

Here

$$Y(\lambda r) = Y_0(\lambda r) - J_0(\lambda r)\left(\ln\frac{\lambda}{2} + C\right)\frac{2}{\pi}$$

$$K(\lambda r) = K_0(\lambda r) + I_0(\lambda r)\left(\ln\frac{\lambda}{2} + \frac{C}{2}\right)$$

$$\left(r = \sqrt{(x-\xi)^2 + (y-\eta)^2}\right)$$

$Y_0(\lambda r)$ and $K_0(\lambda r)$ are, respectively, ordinary and modified Bessel functions of the second kind, and C is Euler's constant. The integration is carried out over the region S in which the distributed load is given.

Bearing in mind that the expression in square brackets under the integral sign in (3.1) for $r \to 0$ has a singularity of the type $r^2 \ln r$, we easily verify that the function $w_0(x, y)$ does satisfy the nonhomogeneous equation (1.1).

The particular solution, taken in the form (3.1), allows a passage to the limit for $\lambda \to 0$. Thus we obtain the particular solution of the nonhomogeneous biharmonic equation

$$w_{01}(x, y) = \frac{1}{8\pi D}\iint_S q(\xi, \eta)\left(r^2 \ln r - \frac{3}{4} r^2\right) d\xi\, d\eta$$

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BIBLIOGRAPHY

1. Sherman, D.I., O poperechnom izgibe plastinki, opertoi vdol kraia, sostavlennogo iz neskol'kikh zamknutykh krivykh (On the transverse

bending of a plate, simply supported along the edge formed from several closed curves). *PMM* Vol. 23, No. 1, 1959.

2. Sherman, D.I., Ob uprugom ravnovesii plastinki, opertoi po kraiu (On the elastic equilibrium of a plate simply supported along the edge). *Izv. Akad. Nauk Arm.SSR, ser. fiz-matem. nauk,* Vol. 10, No. 3, 1957.

3. Vekua, I.N., *Novye metody resheniia ellipticheskikh uravnenii (New methods of solution of elliptic equations).* Gostekhizdat, 1948.

4. Sherman, D.I., Ob ustanovivshikhsia uprugikh kolebaniakh pri zadannykh smeshcheniiakh na granitse sredy (On the steady-state elastic oscillations with given displacements on the boundary of the medium). *PMM* Vol. 10, Nos. 5, 6, 1946.

5. Tamarkin, J.D., On Fredholm's integral equations whose kernels are analytic in a parameter. *Ann. Math.,* Vol. 28, No. 2, 1927.

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